

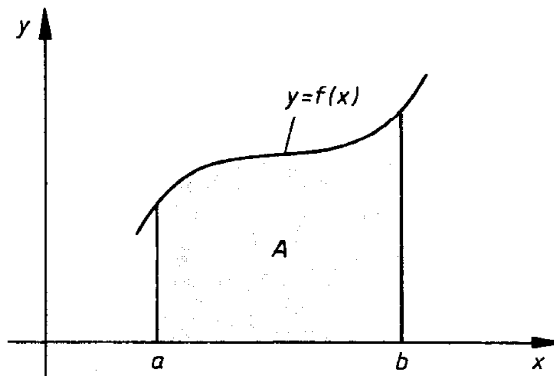
# Definite integral

## Area under a curve

$$f: D \rightarrow \mathbb{R} \quad (D \subseteq \mathbb{R})$$

$$x \mapsto y = f(x)$$

Suppose that  $f(x) \geq 0$  on the interval  $a \leq x \leq b$ :



$A$  = area between the graph of  $f$  and the  $x$ -axis on the interval  $a \leq x \leq b$

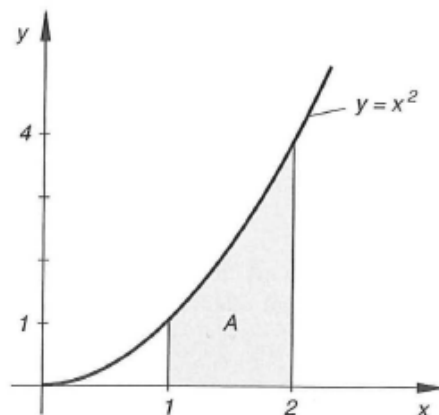
## Definition

The area  $A$  between the graph of  $f$  and the  $x$ -axis on the interval  $a \leq x \leq b$  is the **definite integral** of  $f$  from  $a$  to  $b$ , denoted  $\int_a^b f(x) \, dx$ .

$a$  and  $b$  are called the **limits (or bounds) of integration**.

$$A = \int_a^b f(x) \, dx$$

Ex.:  $f(x) = x^2$



$$A = \int_1^2 x^2 \, dx$$

## Fundamental theorem of calculus

$$\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a) \quad \text{where } F \text{ is any antiderivative of } f$$

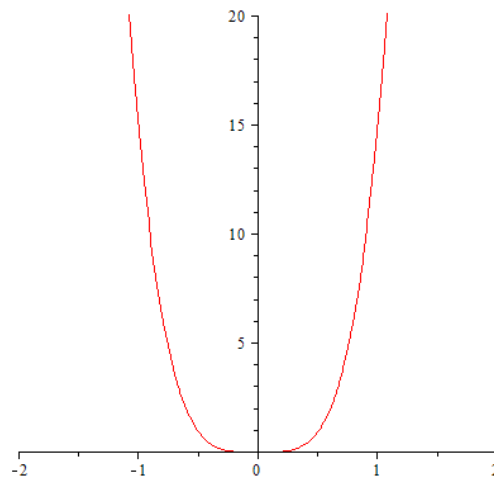
Ex.: 1.  $f(x) = x^2$ ,  $a = 1$ ,  $b = 2$

$$\int_1^2 x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_1^2 = \frac{1}{3} 2^3 - \frac{1}{3} 1^3 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.\bar{3}$$

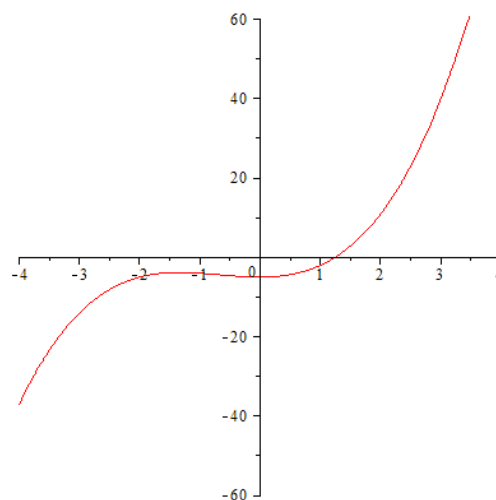
or:  $\int_1^2 x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_1^2 = \frac{1}{3} [x^3]_1^2 = \frac{1}{3} (2^3 - 1^3) = \frac{7}{3} = 2.\bar{3}$

2.  $\int_0^2 x^3 \, dx = \left[ \frac{1}{4} x^4 \right]_0^2 = \frac{1}{4} [x^4]_0^2 = \frac{1}{4} (2^4 - 0^4) = 4$

3.  $\int_{-1}^1 15x^4 \, dx = \left[ 15 \cdot \frac{1}{5} x^5 \right]_{-1}^1 = 3[x^5]_{-1}^1 = 3(1^5 - (-1)^5) = 6$



4.  $\int_2^3 (x^3 + 2x^2 - 5) \, dx = \left[ \frac{1}{4} x^4 + 2 \cdot \frac{1}{3} x^3 - 5x \right]_2^3 = \left( \frac{3^4}{4} + \frac{2 \cdot 3^3}{3} - 5 \cdot 3 \right) - \left( \frac{2^4}{4} + \frac{2 \cdot 2^3}{3} - 5 \cdot 2 \right) = \frac{287}{12} = 23.91\bar{6}$



## Consumer's Surplus

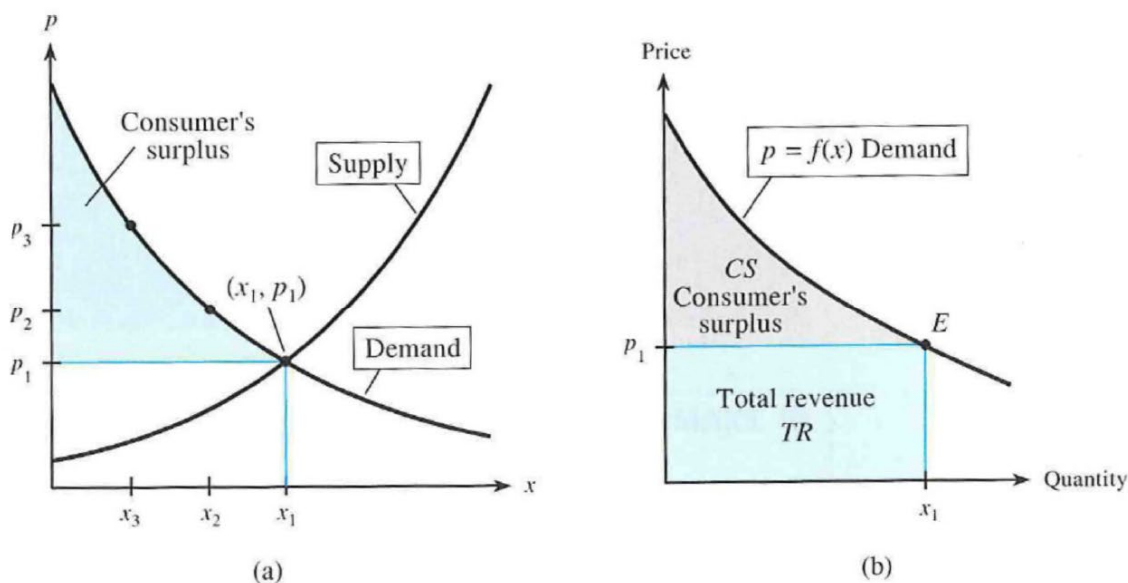
Suppose that the demand for a product is given by  $p = f(x)$  and that the supply of the product is described by  $p = g(x)$ . The price  $p_1$  where the graphs of these functions intersect is the **equilibrium price** (see Figure 13.21(a)). As the demand curve shows, some consumers (but not all) would be willing to pay more than  $p_1$  for the product.

For example, some consumers would be willing to buy  $x_3$  units if the price were  $p_3$ . Those consumers willing to pay more than  $p_1$  are benefiting from the lower price. The total gain for all those consumers willing to pay more than  $p_1$  is called the **consumer's surplus**, and under proper assumptions the area of the shaded region in Figure 13.21(a) represents this consumer's surplus.

Looking at Figure 13.21(b), we see that if the demand curve has equation  $p = f(x)$ , the consumer's surplus is given by the area between  $f(x)$  and the  $x$ -axis from 0 to  $x_1$ , *minus* the area of the rectangle denoted  $TR$ :

$$CS = \int_0^{x_1} f(x) dx - p_1 x_1$$

Note that with equilibrium price  $p_1$  and equilibrium quantity  $x_1$ , the product  $p_1 x_1$  is the area of the rectangle that represents the total dollars spent by consumers and received as revenue by producers (see Figure 13.21(b)).



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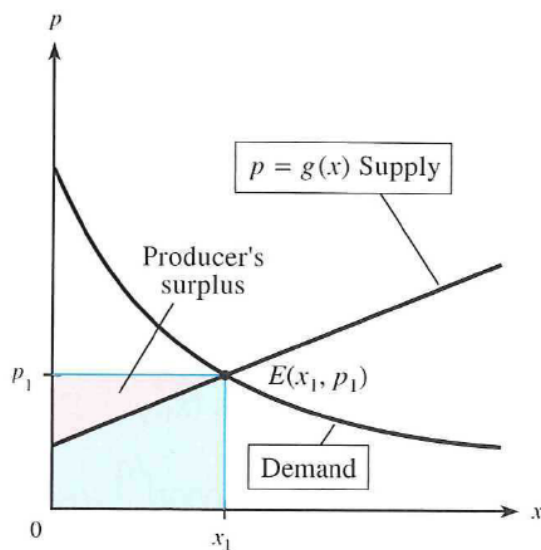
## Producer's Surplus

When a product is sold at the equilibrium price, some producers will also benefit, for they would have sold the product at a lower price. The area between the line  $p = p_1$  and the supply curve (from  $x = 0$  to  $x = x_1$ ) gives the producer's surplus (see Figure 13.23).

If the supply function is  $p = g(x)$ , the **producer's surplus** is given by the area between the graph of  $p = g(x)$  and the  $x$ -axis from 0 to  $x_1$  *subtracted from* the area of the rectangle  $0x_1Ep_1$ .

$$PS = p_1x_1 - \int_0^{x_1} g(x) dx$$

Note that  $p_1x_1$  represents the total revenue at the equilibrium point.



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