

Exercises 13 Derivative

Rate of change, derivative of constant/power/exponential functions

Objectives

- be able to determine a rate of change of a constant/linear function.
- be able to determine a rate of change of a basic power/exponential function.
- be able to estimate a rate of change out of the graph of a function.
- be able to determine the derivative of a constant/linear function.
- be able to determine the derivative of a basic power/exponential function.

Problems

13.1 For each of the following functions $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow y = f(x) = \dots$

- i) ... draw the graph of f .
- ii) ... determine the rate of change $f'(x_0)$ at the given x_0 .

- a) $f(x) = 3$ $x_0 = 2$
- b) $f(x) = c$ ($c \in \mathbb{R}$) any $x_0 \in \mathbb{R}$
- c) $f(x) = 2x - 3$ $x_0 = 4$
- d) $f(x) = mx + q$ ($m \in \mathbb{R} \setminus \{0\}, q \in \mathbb{R}$) any $x_0 \in \mathbb{R}$
- e) * $f(x) = |x|$ any $x_0 \in \mathbb{R}$

13.2 For each of the functions in problem 13.1. ...

- i) ... determine the derivative f' (domain, codomain, equation).
- ii) ... draw the graph of f' .

13.3 Look at the function f and its derivative f' :

$$f: D \rightarrow \mathbb{R}$$
$$x \rightarrow y = f(x) = 24\sqrt{x+1} - 2x - 60$$

$$f': D_1 \rightarrow \mathbb{R}$$
$$x \rightarrow y = f'(x) = \frac{12}{\sqrt{x+1}} - 2$$

Determine the largest possible ...

- a) ... domain D of f .
- b) ... domain D_1 of f' .

13.4 Determine $f'(x)$:

- | | | |
|-------------------------|---------------------------|------------------------------|
| a) $f(x) = 3$ | b) $f(x) = 0$ | c) $f(x) = -1$ |
| d) $f(x) = x^3$ | e) $f(x) = x^4$ | f) $f(x) = x^5$ |
| g) $f(x) = x^{17}$ | h) $f(x) = x^{200}$ | i) $f(x) = x^{100'001}$ |
| j) $f(x) = x^{-1}$ | k) $f(x) = x^{-2}$ | l) $f(x) = x^{-17}$ |
| m) $f(x) = \frac{1}{x}$ | n) $f(x) = \frac{1}{x^3}$ | o) $f(x) = \frac{1}{x^{99}}$ |

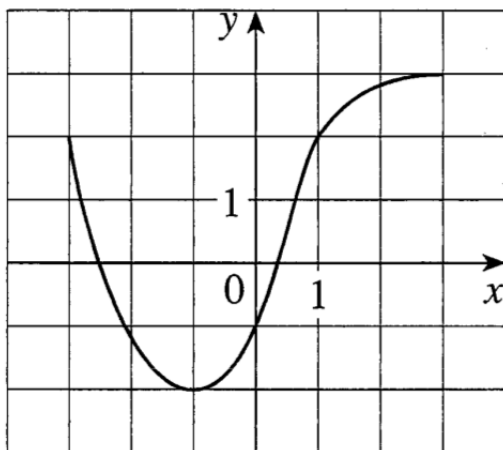
13.5 Determine $f'(x)$:

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|--|--|--|
| a) $f(x) = 3^x$ | b) $f(x) = 5^x$ | c) $f(x) = 18^x$ |
| d) $f(x) = \left(\frac{2}{3}\right)^x$ | e) $f(x) = \left(\frac{13}{17}\right)^x$ | f) $f(x) = \left(\frac{1}{4}\right)^x$ |
| g) $f(x) = \left(\frac{1}{e}\right)^x$ | h) $f(x) = \left(\frac{3}{e}\right)^x$ | i) $f(x) = \left(\frac{e}{3}\right)^x$ |

13.6 Determine the rate of change of the function f at the indicated values of x :

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|--|------------------------|-------------------------|
| a) $f(x) = x$ | | |
| i) $x = 0$ | ii) $x = 1$ | iii) $x = -2$ |
| b) $f(x) = x^5$ | | |
| i) $x = 0$ | ii) $x = 2$ | iii) $x = -\frac{2}{3}$ |
| c) $f(x) = x^{-4}$ | | |
| i) $x = -1$ | ii) $x = -\frac{4}{3}$ | iii) $x = 0$ |
| d) $f(x) = \left(\frac{2}{3}\right)^x$ | | |
| i) $x = 0$ | ii) $x = 1$ | iii) $x = -2$ |

13.7 The graph of a function f is given as follows:



Estimate the rate of change $f'(x_0)$ at the given x_0 :

- | | |
|---------------|---------------|
| a) $x_0 = -1$ | b) $x_0 = 0$ |
| c) $x_0 = 1$ | d) $x_0 = -2$ |

Hints:

- Draw the tangent to the graph of f at the given x_0 .
- Estimate the slope of the tangent.

13.8 * The rate of change $f'(x_0)$ of f at x_0 can be determined by looking at the secant through the points $A(x_0 | f(x_0))$ and $B(x_0 + \Delta x | f(x_0 + \Delta x))$ of the graph of f . The slope of this secant tends towards the slope of the tangent at $A(x_0 | f(x_0))$ as Δx tends towards 0.

It has been shown in class how to find $f'(x_0)$ in that way for the quadratic function $f(x) = x^2$.

Find $f'(x_0)$ for the following functions f :

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|-----------------|---------------------------|
| a) $f(x) = x^3$ | b) $f(x) = \frac{1}{x^2}$ |
|-----------------|---------------------------|

Answers

- 13.1 a) i) ...
 ii) $f'(2) = 0$
 b) i) ...
 ii) $f'(x_0) = 0$
 c) i) ...
 ii) $f'(4) = 2$
 d) i) ...
 ii) $f'(x_0) = m$
 e) * i) ...
 ii) $f'(x_0) = \begin{cases} 1 & (x_0 > 0) \\ -1 & (x_0 < 0) \\ \text{not defined} & (x_0 = 0) \end{cases}$

- 13.2 a) i) $f': \mathbb{R} \rightarrow \mathbb{R}$
 $x \rightarrow y = f'(x) = 0$
 ii) ...
 b) i) $f': \mathbb{R} \rightarrow \mathbb{R}$
 $x \rightarrow y = f'(x) = 0$
 ii) ...
 c) i) $f': \mathbb{R} \rightarrow \mathbb{R}$
 $x \rightarrow y = f'(x) = 2$
 ii) ...
 d) i) $f': \mathbb{R} \rightarrow \mathbb{R}$
 $x \rightarrow y = f'(x) = m$
 ii) ...
 e) * i) $f': \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$
 $x \rightarrow y = f'(x) = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases}$
 ii) ...

- 13.3 a) $D = \{x: x \in \mathbb{R} \text{ and } x \geq -1\}$
 b) $D_1 = \{x: x \in \mathbb{R} \text{ and } x > -1\}$

Hints:

- The square root of a negative number is not defined.
- Division by zero is not defined.

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|------|-----------------------------|-----------------------------|----------------------------------|
| 13.4 | a) $f'(x) = 0$ | b) $f'(x) = 0$ | c) $f'(x) = 0$ |
| | d) $f'(x) = 3x^2$ | e) $f'(x) = 4x^3$ | f) $f'(x) = 5x^4$ |
| | g) $f'(x) = 17x^{16}$ | h) $f'(x) = 200x^{199}$ | i) $f'(x) = 100'001x^{100'000}$ |
| | j) $f'(x) = -x^{-2}$ | k) $f'(x) = -2x^{-3}$ | l) $f'(x) = -17x^{-18}$ |
| | m) $f'(x) = -\frac{1}{x^2}$ | n) $f'(x) = -\frac{3}{x^4}$ | o) $f'(x) = -\frac{99}{x^{100}}$ |

- 13.5 a) $f'(x) = 3^x \ln(3)$ b) $f'(x) = 5^x \ln(5)$ c) $f'(x) = 18^x \ln(18)$
 d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$ e) $f'(x) = \left(\frac{13}{17}\right)^x \ln\left(\frac{13}{17}\right)$
 f) $f'(x) = \left(\frac{1}{4}\right)^x \ln\left(\frac{1}{4}\right) = -\frac{\ln(4)}{4^x}$
 Hint:
 - Logarithm rules (see formulary) can be applied in order to simplify the result.
 g) $f'(x) = -\frac{1}{e^x}$ h) $f'(x) = \left(\frac{3}{e}\right)^x (\ln(3) - 1)$ i) $f'(x) = \left(\frac{e}{3}\right)^x (1 - \ln(3))$
- 13.6 a) $f'(x) = 1$
 i) $f'(0) = 1$ ii) $f'(1) = 1$ iii) $f'(-2) = 1$
 b) $f'(x) = 5x^4$
 i) $f'(0) = 0$ ii) $f'(2) = 80$ iii) $f'\left(-\frac{2}{3}\right) = \frac{80}{81}$
 c) $f'(x) = -\frac{4}{x^5}$
 i) $f'(-1) = 4$ ii) $f'\left(-\frac{4}{3}\right) = \frac{243}{256}$ iii) $f'(0)$ is not defined
 d) $f'(x) = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)$
 i) $f'(0) = \ln\left(\frac{2}{3}\right)$ ii) $f'(1) = \frac{2}{3} \ln\left(\frac{2}{3}\right)$ iii) $f'(-2) = \frac{9}{4} \ln\left(\frac{2}{3}\right)$
- 13.7 a) $f'(-1) \approx 0$ b) $f'(0) \approx 2$
 c) $f'(1) \approx \frac{3}{2}$ d) $f'(-2) \approx -\frac{5}{3}$
- 13.8 * a) ... b) ...