Review exercises 2 Differential calculus, integral calculus

Problems

R2.1 Decide whether the following statements are true or false:

a) "The derivative of a function is a function."

b) "The derivative of a function at a particular value of the variable is a number."

c) "The function f has a relative maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."

d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."

e) "Suppose that the function f has a relative maximum at $x = x_1$. If there are no other relative maxima and if there is no relative minimum at all, then the relative maximum is the absolute maximum of f."

f) "If g' = f , then g is an antiderivative of f."

g) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."

h) "f with $f(x) = 3x$ has infinitely many antiderivatives."

i) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ at x_0 for the following functions f:

a) $f(x) = 4x^2(x^2 - 1)$

i) $x_0 = 0$ ii) $x_0 = -1$

b) $f(x) = (- 3x^2 + 2x - 1)\cdot e^x$

i) $x_0 = 0$ ii) $x_0 = -2$

c) $f(x) = (x^2 + 2)\cdot e^{-3x}$

i) $x_0 = 1$ ii) $x_0 = -\frac{1}{3}$

R2.3 For the given cost function C(x) and revenue function R(x) determine ...

i) ... the marginal cost function C'(x).

ii) ... the marginal revenue function R'(x).

iii) ... the marginal profit function P'(x).

a) $C(x) = 200 + 40x$ $R(x) = 60x$

b) $C(x) = 100 + 20x + 5x^2$ $R(x) = 100x - 2x^2$

c) $C(x) = 50 + 20x^2 + 3e^{4x}$ $R(x) = 200x - e^{-4x^2}$

R2.4 For each function, find …

i) … the relative maxima and minima.

ii) … the points of inflection.

a) $f(x) = 2x^3 - 9x^2 + 12x - 1$

b) f(x) as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$R(x) = 36x - 0.01x^2$

Find the maximum revenue …

a) ... if production is not limited to a certain number of units.

b) ... if production is limited to at most 1500 units.

R2.6 If the total cost function for a product is

$C(x) = 100 + x^2$

producing how many units x will result in a minimum average cost per unit? Find the minimum average cost.

R2.7 A firm can produce only 1000 units per month. The monthly total cost ist given by

$C(x) = 300 + 200x$

dollars, where x is the number produced. If the total revenue is given by

$R(x) = 250x - \frac{1}{100}x^2$

dollars, how many items should the firm produce for maximum profit? Find the maximum profit.

R2.8 Determine the indefinite integrals below:

a) $\int \left(x^4 - 3x^3 - 6\right)\, dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\, dx$

R2.9 The equation of the third derivative f''' of a function f is given as follows:

$f'''(x) = 3x + 1$

Find the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.10 If the marginal cost (in dollars) for producing a product is $C'(x) = 5x + 10$, with a fixed cost of \$800, what will be the cost of producing 20 units?

R2.11 A certain firm's marginal cost $C'(x)$ and the derivative of the average revenue $\overline{R}'(x)$ are given as follows:

$C'(x) = 6x + 60$

$\overline{R}'(x) = -1$

The total cost and revenue of the production of 10 items are \$1000 and \$1700, respectively.

How many units will result in a maximum profit? Find the maximum profit.

R2.12 The demand function for a product is

$p = f(x) = 49 - x^2$

and the supply function is

$p = g(x) = 4x + 4$

Find the equilibrium point and both the consumer’s and the producer’s surplus there.

R2.13 The demand function for a product is

$$p = f(x) = 110 - ax^2$$

and the supply function is

$$p = g(x) = 2 - \frac{6}{5}x + bx^2$$

with unknown parameters a and b. The equilibrium price is \$10, and the producer's surplus is \$73.33

Determine the two unknown parameters a and b.

Answers

R2.1
a) true b) true c) false
d) true e) true f) true
g) false h) true i) true

R2.2 a) $f'(x) = 16x^3 - 8x$
$f''(x) = 48x^2 - 8$
i) $f(0) = 0$ $f'(0) = 0$ $f''(0) = -8$
ii) $f(-1) = 0$ $f'(-1) = -8$ $f''(-1) = 40$

b) $f'(x) = (-3x^2 - 4x + 1)\cdot e^x$
$f''(x) = (-3x^2 - 10x - 3)\cdot e^x$
i) $f(0) = -1$ $f'(0) = 1$ $f''(0) = -3$
ii) $f(-2) = -17\cdot e^{-2} = -2.300\ldots$
$f'(-2) = -3\cdot e^{-2} = -0.406\ldots$
$f''(-2) = 5\cdot e^{-2} = 0.676\ldots$

c) $f'(x) = (-3x^2 + 2x - 6)\cdot e^{-3x}$
$f''(x) = (9x^2 - 12x + 20)\cdot e^{-3x}$
i) $f(1) = 3\cdot e^{-3} = 0.149\ldots$
$f'(1) = -7\cdot e^{-3} = -0.348\ldots$
$f''(1) = 17\cdot e^{-3} = 0.846\ldots$
ii) $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\ldots$
$f'\left(-\frac{1}{3}\right) = -7e = -19.027\ldots$
$f''\left(-\frac{1}{3}\right) = 25e = 67.957\ldots$

R2.3 a) i) $C'(x) = 40$ ii) $R'(x) = 60$
iii) $P'(x) = 20$

b) i) $C'(x) = 20 + 10x$ ii) $R'(x) = 100 - 4x$
iii) $P'(x) = 80 - 14x$

c) i) $C'(x) = 40x + 12e^{4x}$ ii) $R'(x) = 200 + 8x\, e^{-4x^2}$
iii) $P'(x) = 200 - 40x - 12e^{4x} + 8x\, e^{-4x^2}$

R2.4 a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
$f'(x) = 6x^2 - 18x + 12$
$f''(x) = 12x - 18$
$f'''(x) = 12$
i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
$f''(x_1) = -6 < 0$ $\Rightarrow$ relative maximum at $x_1 = 1$
$f''(x_2) = 6 > 0$ $\Rightarrow$ relative minimum at $x_2 = 2$

ii) $f''(x) = 0$ at $x_3 = \frac{3}{2}$
$f'''(x_3) = 12 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{3}{2}$

b) $f(x) = 4x^2(x^2 - 1)$
$f'(x) = 16x^3 - 8x = 8x(2x^2 - 1)$
$f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$
$f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
$f''(x_1) = -8 < 0$ $\Rightarrow$ relative maximum at $x_1 = 0$
$f''(x_2) = 16 > 0$ $\Rightarrow$ relative minimum at $x_2 = \frac{1}{\sqrt{2}}$
$f''(x_3) = 16 > 0$ $\Rightarrow$ relative minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{\sqrt{6}}$
$f'''(x_3) = \frac{96}{\sqrt{6}} \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{1}{\sqrt{6}}$

R2.5 a) **Relative** maximum at $x_1 = 1800$
$R(x_1) = \$32'400$
$R(x) < R(x_1)$ if $x \neq x_1$ as there is no relative minimum
$\Rightarrow$ R = \$32'400 is the **absolute** maximum revenue at x = 1800.

b) Relative maximum at x = 1800 lies outside the possible interval $0 \leq x \leq 1500$
R(1500) = \$31'500 > R(0) = 0\$
$\Rightarrow$ R = \$31'500 is the **absolute** maximum revenue at x = 1500.

R2.6 $\overline{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$
$\overline{C}(x)$ has a **relative** minimum at $x_1 = 10$
$\overline{C}(20) = \$20$
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
$\Rightarrow$ $\overline{C}$ = \$20 is the **absolute** minimum average cost at x = 10.

R2.7 $P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$
P(x) has a **relative** maximum at $x_1 = 2500$. This is outside the possible interval $0 \leq x \leq 1000$
P(1000) = \$39'700 > P(0) = - 300\$
$\Rightarrow$ P = \$39'700 is the **absolute** maximum profit at the endpoint x = 1000.

R2.8 a) $\int \left(x^4 - 3x^3 - 6\right)\, dx = \frac{x^5}{5} + \frac{3x^4}{4} - 6x + C$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\, dx = \frac{x^7}{14} + \frac{2}{9x^3} + C$

R2.9 $f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$

R2.10 C(20) = \$2000

Hint:
- First, determine the cost function C(x) $\Rightarrow$ $C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.11 P = \$800 is the absolute maximum profit at x = 15 units.

Hints:
- Determine the cost function $C(x) \Rightarrow C(x) = 3x^2 + 60x + 100$
- Determine the average revenue function $\overline{R}(x) \Rightarrow \overline{R}(x) = - x + C$
- Determine the revenue function $R(x) \Rightarrow R(x) = - x^2 + 180x$
- Find the profit function $P(x) \Rightarrow P(x) = - 4x^2 + 120x - 100$
- Find the relative maximum of the profit function P(x).
- Check if the relative maximum is the absolute maximum.

R2.12

Equilibrium quantity	x = 5
Equilibrium price	p = 24
Consumer's surplus	CS = 83.33
Producer's surplus	PS = 50

R2.13 a = 1
b = 0.2