Exercises 14 Differentiation rules
Coefficient/sum/product rule, chain rule, higher-order derivatives

Objectives

- be able to apply the coefficient, sum, product rule to determine the derivative of a function.
- be able to apply the chain rule to determine the derivative of a function.
- be able to determine a higher-order derivative of a function.

Problems

14.1 Determine the derivative by applying the **coefficient rule**:

a) $f(x) = 3x^5$ b) $f(x) = -4x^3$ c) $f(x) = -x^{10}$

d) $f(x) = a \cdot x^3$ e) $f(x) = n \cdot x^{n-1}$ f) $f(x) = 9 \cdot 3^x$

g) $s(t) = \frac{1}{2} g \cdot t^2$ h) $S(T) = \alpha \cdot T^4$ i) $C(x) = (-3x)^3$

14.2 Determine the derivative by applying the **sum rule**:

a) $f(x) = x^5 + x^6$ b) $f(x) = x^{10} - x^9$ c) $f(x) = 1 + x + 3x^3$

d) $f(x) = \frac{1}{4} x^4 + 3x^2 - 2$ e) $f(x) = 3x^2(x - 2)$ f) $f(x) = - 3x^8 + x^5 - 3x + 99$

g) $f(x) = ax^2 + bx + c$ h) $f(x) = 3(a^2 - 2ax + x^2)$ i) $f(x) = \frac{x^3}{3} - \frac{3}{x^3}$

j) $s(t) = s_0 + v_0 t + \frac{1}{2} g \cdot t^2$ k) $V(r) = - \frac{a}{r} + \frac{b}{r^2}$ l) $C(n) = C_0(1 + nr)$

14.3 Determine the derivative by applying the **product rule**:

a) $f(x) = x \cdot e^x$ b) $f(x) = x^3 \cdot 3^x$

c) $f(x) = - 2x^5(x - 1)$ d) $f(x) = (2x - 1) \cdot e^x$

e) $f(x) = (2x - 1) (- 3x^2 - x + 1)$ f) $f(x) = 3(1 - x^2)(x^{10} - x^9)$

g) $V(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right)$ h) $T(V) = \frac{1}{n \cdot R} \left(p + \frac{a \cdot n^2}{V^2}\right) (V - n \cdot b)$

14.4 Determine the derivative by applying the **chain rule**:

a) $f(x) = (2x)^3$ b) $f(x) = (3x - 1)^5$ c) $f(x) = (- 3x^3 + x^2 - 4x)^6$

d) $f(x) = e^{4x}$ e) $f(x) = e^{-x}$ f) $f(x) = e^{1 - \frac{x}{2}}$

g) $f(x) = e^{-x^2}$ h) $f(x) = e^{x^2-2x+5}$ i) $f(x) = e^{e^x}$

j) $f(x) = 2^{3^x}$ k) * $f(x) = 2^{e^{2x}}$ l) ** $f(x) = x^x$

14.5 Determine the derivative by applying the appropriate differentiation rule(s), and simplify the expression as far as possible:

a) $f(x) = (x - 2) e^{2x}$ b) $f(x) = (2 - x^2) e^{-x}$

c) $f(x) = (3x^3 - 2x^2 + x - 1) e^{-2x}$ d) $f(x) = (x - 2)^2 e^{- x^2 - 2x}$

e) $f(x) = ax\, e^{-\frac{x^2}{2}}$ f) $P(v) = av^2 e^{- bv^2}$

14.6 Determine the derivative of the indicated function at the indicated value of the variable:

a)	f in 14.1 b)	$x = 2$	b)	s in 14.1 g)	$t = 4$
c)	f in 14.2 g)	$x = -1$	d)	f in 14.5 e)	$x = 0$

14.7 Determine the second and third derivatives of the functions in problem …

a)	… 14.1 a)	b)	… 14.2 g)
c)	… 14.3 a)	d)	… 14.4 g)
e)	… 14.5 b)	f)	… 14.5 e)

14.8 Determine the indicated higher-order derivatives:

a) $f''(-1)$ with function f in 14.1 a)

Hint:
- You have already determined $f''(x)$ in 14.7 a).

b) $f'''(2)$ with function f in 14.5 e)

Hint:
- You have already determined $f'''(x)$ in 14.7 f).

14.9 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The third derivative of a function is a ...

- ☐ ... constant function if the second derivative is a quadratic function.
- ☐ ... quadratic function if the second derivative is a linear function.
- ☐ ... linear function if the first derivative is a quadratic function.
- ☐ ... constant function if the first derivative is a quadratic function.

b) The derivative of a ...

- ☐ ... product is the product of the derivatives of the single factors.
- ☐ ... sum is the sum of the derivatives of the single factors.
- ☐ ... composite function is the sum of the two composite functions.
- ☐ ... constant is the constant itself.

c) If $f(x) = c \cdot g(x) \cdot h(x)$ then $f'(x) = ...$

- ☐ ... 0
- ☐ ... $c \cdot g'(x) \cdot h'(x)$
- ☐ ... $c \cdot g(x) \cdot h'(x) + c \cdot g'(x) \cdot h(x)$
- ☐ ... $c \cdot g'(x) \cdot h'(x) + c \cdot g(x) \cdot h(x)$

Answers

14.1
a) $f'(x) = 3 \cdot 5x^4 = 15x^4$
b) $f'(x) = (-4)\, 3x^2 = -12x^2$
c) $f'(x) = (-1)\, 10x^9 = -10x^9$
d) $f'(x) = a \cdot 3x^2 = 3ax^2$

Hint:
- a is a constant.

e) $f'(x) = n(n-1)x^{n-2}$
f) $f'(x) = 9 \cdot 3^x \cdot \ln(3)$
g) $s'(t) = \frac{g}{2}\, 2t = gt$

Hints:
- The name of the function is s, and the variable is t.
- g is a constant.

h) $S'(T) = \alpha \cdot 4T^3 = 4\alpha T^3$
i) $C'(x) = -\, 81x^2$

14.2
a) $f'(x) = 5x^4 + 6x^5$
b) $f'(x) = 10x^9 - 9x^8$
c) $f'(x) = 1 + 9x^2$
d) $f'(x) = x^3 + 6x$
e) $f'(x) = 9x^2 - 12x$
f) $f'(x) = -\, 24x^7 + 5x^4 - 3$
g) $f'(x) = 2ax + b$
h) $f'(x) = -\, 6a + 6x$
i) $f'(x) = x^2 + \frac{9}{x^4}$
j) $s'(t) = v_0 + gt$
k) $V'(r) = \frac{a}{r^2} - \frac{2b}{r^3}$
l) $C'(n) = C_0 \cdot r$

14.3
a) $f'(x) = e^x + x \cdot e^x$
b) $f'(x) = 3x^2 \cdot 3^x + x^3 \cdot 3^x \cdot \ln(3)$
c) $f'(x) = -\, 2(5x^4(x - 1) + x^5)$
d) $f'(x) = 2 \cdot e^x + (2x - 1) \cdot e^x$
e) $f'(x) = 2(-\, 3x^2 - x + 1) + (2x - 1)\, (-\, 6x - 1)$
f) $f'(x) = 3(-\, 2x(x^{10} - x^9) + (1 - x^2)\, (10x^9 - 9x^8))$
g) $V'(r) = e^r \left(a \cdot r^2 - \frac{b}{r^3}\right) + e^r \left(2a \cdot r + \frac{3b}{r^4}\right)$

Hints:
- V is the name of the function, and r is the variable.
- a and b are constants.

h) $T'(V) = \frac{1}{n \cdot R}\left(-\frac{2a \cdot n^2}{V^3}\,(V - n \cdot b) + \left(p + \frac{a \cdot n^2}{V^2}\right)\right)$

Hints:
- T is the name of the function, and V is the variable.
- n, R, p, a and b are constants.

14.4
a) $f'(x) = 3(2x)^2 \cdot 2 = 24x^2$
b) $f'(x) = 5(3x - 1)^4 \cdot 3 = 15(3x - 1)^4$
c) $f'(x) = 6(-\, 3x^3 + x^2 - 4x)^5 \cdot (-\, 9x^2 + 2x - 4)$
d) $f'(x) = e^{4x}\, 4 = 4\, e^{4x}$
e) $f'(x) = e^{-x}\, (-1) = -\, e^{-x}$
f) $f'(x) = e^{1 - \frac{x}{2}} \left(-\frac{1}{2}\right) = -\frac{1}{2}\, e^{1 - \frac{x}{2}}$
g) $f'(x) = e^{-x^2}(-2x) = -\, 2x \cdot e^{-x^2}$
h) $f'(x) = e^{x^2 - 2x + 5}(2x - 2)$
i) $f'(x) = e^{e^x} \cdot e^x$
j) $f'(x) = 2^{3^x} \cdot \ln(2) \cdot 3^x \cdot \ln(3)$

k) * $f'(x) = 2^{e^{2x}} \cdot \ln(2) \cdot e^{2x} \cdot 2$

l) ** $f'(x) = x^x \cdot (\ln(x) + 1)$

Hints:
- The expression x^x can be rewritten as follows: $x^x = e^{\ln(x^x)} = e^{x \cdot \ln(x)}$
- The derivative of $\ln(x)$ is $\frac{1}{x}$

14.5

a) $f'(x) = e^{2x} + (x - 2)\, e^{2x}\, 2 = (2x - 3)\, e^{2x}$

b) $f'(x) = -2x\, e^{-x} + (2 - x^2)\, e^{-x}\, (-1) = (x^2 - 2x - 2)\, e^{-x}$

c) $f'(x) = (9x^2 - 4x + 1)\, e^{-2x} - 2(3x^3 - 2x^2 + x - 1)\, e^{-2x} = (-6x^3 + 13x^2 - 6x + 3)\, e^{-2x}$

d) $f'(x) = 2(x - 2)e^{-x^2-2x} + (x - 2)^2(-2x - 2)e^{-x^2-2x} = -2\,(x^3 - 3x^2 - x + 6)\, e^{-x^2-2x}$

e) $f'(x) = a\left(e^{-\frac{x^2}{2}} + x\, e^{-\frac{x^2}{2}}(-x)\right) = a(1 - x^2)e^{-\frac{x^2}{2}}$

f) $P'(v) = a\left(2v\, e^{-bv^2} + v^2 e^{-bv^2}(-2bv)\right) = 2av(1 - bv^2)e^{-bv^2}$

14.6

a) $f'(2) = -48$ b) $s'(4) = 4g$

c) $f'(-1) = -2a + b$ d) $f'(0) = a$

14.7

a) 14.1 a)
$f''(x) = 15 \cdot 4x^3 = 60x^3$
$f'''(x) = 60 \cdot 3x^2 = 180x^2$

b) 14.2 g)
$f''(x) = 2a \cdot 1 = 2a$
$f'''(x) = 0$

c) 14.3 a)
$f''(x) = e^x + (e^x + x \cdot e^x) = (x + 2)\, e^x$
$f'''(x) = e^x + (x + 2)\, e^x = (x + 3)\, e^x$

d) 14.4 g)
$f''(x) = -2\left(e^{-x^2} + x\, e^{-x^2}(-2x)\right) = 2(2x^2 - 1)e^{-x^2}$
$f'''(x) = 2\left(4x\, e^{-x^2} + (2x^2 - 1)e^{-x^2}(-2x\right) = 4x(-2x^2 + 3)e^{-x^2}$

e) 14.5 b)
$f''(x) = (2x - 2)e^{-x} + (x^2 - 2x - 2)\, e^{-x}\, (-1) = (4x - x^2)\, e^{-x}$
$f'''(x) = (4 - 2x)\, e^{-x} + (4x - x^2)\, e^{-x}\, (-1) = (x^2 - 6x + 4)\, e^{-x}$

f) 14.5 e)
$f''(x) = a\left(-2x\, e^{-\frac{x^2}{2}} + (1 - x^2)\, e^{-\frac{x^2}{2}}(-x)\right) = a(x^3 - 3x)\, e^{-\frac{x^2}{2}}$
$f'''(x) = a\left((3x^2 - 3)\, e^{-\frac{x^2}{2}} + (x^3 - 3x)\, e^{-\frac{x^2}{2}}(-x)\right) = a(-x^4 + 6x^2 - 3)\, e^{-\frac{x^2}{2}}$

14.8

a) $f''(-1) = -60$

b) $f'''(2) = a(-16 + 6 \cdot 4 - 3)\, e^{-\frac{4}{2}} = \frac{5a}{e^2}$

14.9

a) 4th statement

b) 2nd statement

c) 3rd statement