Exercises 7 Quadratic function and equations
Quadratic function

Objectives

- be able to graph a quadratic function out of the vertex form of its equation.
- be able to determine the position of the vertex of a parabola out of the vertex form of the equation of the corresponding quadratic function.
- be able to convert the vertex form of the equation of a quadratic function into the general form.
- know, understand, and be able to apply the method of completing the square.
- be able to convert the general form of the equation of a quadratic function into the vertex form.

Problems

7.1 Look at the easiest case of a quadratic function:

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y = f(x) = x^2$$

a) Establish a table of values of f for the interval $-4 \leq x \leq 4$.

b) Draw the graph of f in the interval $-4 \leq x \leq 4$ into a Cartesian coordinate system.

7.2 Generally, the equation of a quadratic function can be written in the so-called vertex form below:

$$f: D \to \mathbb{R} \qquad (D \subseteq \mathbb{R})$$
$$x \mapsto y = f(x) = a(x-u)^2 + v \qquad (a \in \mathbb{R}\backslash\{0\},\ u \in \mathbb{R},\ v \in \mathbb{R})$$

In problems a) to d) ...

i) ... sketch the graphs of the functions f_0, f_1, and f_2 into one coordinate system.

ii) ... describe the influence of the parameters u, v, and a on the graph of the quadratic function.

a) Variation of parameter u (in all three cases below: a = 1, v = 0)

$u = 0$: $y = f_0(x) = x^2$
$u = 2$: $y = f_1(x) = (x-2)^2$
$u = -1$: $y = f_2(x) = (x+1)^2$

b) Variation of parameter v (in all three cases below: a = 1, u = 0)

$v = 0$: $y = f_0(x) = x^2$
$v = 3$: $y = f_1(x) = x^2 + 3$
$v = -2$: $y = f_2(x) = x^2 - 2$

c) Variation of parameter a (in all three cases below: u = 0, v = 0)

$a = 1$: $y = f_0(x) = x^2$
$a = 2$: $y = f_1(x) = 2x^2$
$a = -2$: $y = f_2(x) = -2x^2$

d) Variation of parameter a (in all three cases below: u = 0, v = 0)

$a = 1$: $y = f_0(x) = x^2$
$a = \frac{1}{2}$: $y = f_1(x) = \frac{1}{2}x^2$
$a = -\frac{1}{2}$: $y = f_2(x) = -\frac{1}{2}x^2$

7.3 For each quadratic function f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x)$ in a) to h) ...

i) ... state the parameters a, u, and v.

ii) ... state the coordinates of the vertex of the graph.

iii) ... state whether the parabola, i.e. the graph of the function, opens upwards or downwards.

iv) ... graph the function.

a) $y = f(x) = (x + 2)^2$
b) $y = f(x) = -3x^2$
c) $y = f(x) = 2x^2 - 1$
d) $y = f(x) = -(x - 3)^2 + 4$
e) $y = f(x) = \frac{1}{2}(x + 3)^2 + 2$
f) $y = f(x) = -2(x - 1)^2 + 5$
g) $y = f(x) = \frac{5}{2} - \left(x - \frac{1}{2}\right)^2$
h) $y = f(x) = -\frac{1}{2} - 3(2 - x)^2$

7.4 The equation of a quadratic function can be written in the two forms below:

$y = f(x) = ax^2 + bx + c$ general form

$y = f(x) = a(x - u)^2 + v$ vertex form

a) Verify that the vertex form of the equation can always be converted into the general form.

b) Assume that the values of the parameters a, u, and v are known.
Use the result in a) to determine the values of the parameters b and c out of a, u, and v.

7.5 The equation of a quadratic function f is written in the vertex form. Determine the general form of the equation:

a) $y = f(x) = 2(x - 3)^2 + 4$
b) $y = f(x) = -(x + 2)^2 - 3$
c) $y = f(x) = x^2 + 5$
d) $y = f(x) = -3(x - 4)^2$

7.6 Convert the given equation of a quadratic function into the vertex form by completing the square:

a) $y = f(x) = 3x^2 - 12x + 8$
b) $y = f(x) = x^2 + 6x$
c) $y = f(x) = x^2 - 2x + 1$
d) $y = f(x) = 2x^2 + 12x + 18$
e) $y = f(x) = -2x^2 - 6x - 2$
f) $y = f(x) = x^2 + 1$
g) $y = f(x) = -\frac{1}{2}x^2 + 2x - 2$
h) $y = f(x) = -4x^2 + 24x - 43$
i) $y = f(x) = 2(x - 3)(x + 4)$
j) $y = f(x) = x + 3 - \left(x + \frac{1}{2}\right) x$

7.7 For the graphs of the quadratic functions f: $\mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x)$ in a) to f) ...

i) ... determine the coordinates of the vertex.

ii) ... state whether the parabola opens upwards or downwards.

a) $y = f(x) = 2x^2 + 12x + 20$
b) $y = f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{1}{2}$
c) $y = f(x) = 12x - 3x^2 - 11$
d) $y = f(x) = x(-0.2x + 1.2) - 2.8$
e) $y = f(x) = \frac{17 + 12x + 2x^2}{4}$
f) $y = f(x) = 7x(3 - x) - 13.25$

7.8 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The graph of a quadratic function ...

- ☐ ... always intersects the x-axis in two points.
- ☐ ... opens downwards if it has no point in common with the x-axis.
- ☐ ... touches the x-axis if there is only one vertex.
- ☐ ... is always a parabola.

b) f is a linear function, and g is a quadratic function. It can be concluded that the graphs of f and g ...

- ☐ ... have no points in common.
- ☐ ... intersect only if the slope of f is not equal to zero.
- ☐ ... cannot have more than two points in common.
- ☐ ... have at least one point in common.

c) The vertex form of the equation of a quadratic function ...

- ☐ ... is equal to the general form if the vertex of the graph is on the y-axis.
- ☐ ... can be obtained from the general form by multiplying out all the terms.
- ☐ ... does not exist if the graph opens downwards.
- ☐ ... only depends on the position of the vertex.

Answers

7.1 see theory

7.2 see theory

7.3 a) i) $a = 1$, $u = -2$, $v = 0$
ii) $V(-2|0)$
iii) parabola opens upwards
iv) ...

b) i) $a = -3$, $u = 0$, $v = 0$
ii) $V(0|0)$
iii) parabola opens downwards
iv) ...

c) i) $a = 2$, $u = 0$, $v = -1$
ii) $V(0|-1)$
iii) parabola opens upwards
iv) ...

d) i) $a = -1$, $u = 3$, $v = 4$
ii) $V(3|4)$
iii) parabola opens downwards
iv) ...

e) i) $a = \frac{1}{2}$, $u = -3$, $v = 2$
ii) $V(-3|2)$
iii) parabola opens upwards
iv) ...

f) i) $a = -2$, $u = 1$, $v = 5$
ii) $V(1|5)$
iii) parabola opens downwards
iv) ...

g) i) $a = -1$, $u = \frac{1}{2}$, $v = \frac{5}{2}$
ii) $V\left(\frac{1}{2}\middle|\frac{5}{2}\right)$
iii) parabola opens downwards
iv) ...

h) i) $a = -3,\ u = 2,\ v = -\frac{1}{2}$

ii) $V\left(2|-\frac{1}{2}\right)$

iii) parabola opens downwards

iv) ...

7.4 a) $y = f(x) = a(x - u)^2 + v = ... = ax^2 - 2aux + au^2 + v = ax^2 + (-2au)x + (au^2 + v)$

Hints:
- Expand the term $(x-u)^2$.
- Simplify the whole expression.

b) $b = -2au$
$c = au^2 + v$

Hint:
- Compare the resulting expression in a) with the general form $ax^2 + bx + c$.

7.5 a) $y = f(x) = 2x^2 - 12x + 22$

b) $y = f(x) = -x^2 - 4x - 7$

c) $y = f(x) = x^2 + 5$

Notice:
- This is both the general and the vertex form of the equation.

d) $y = f(x) = -3x^2 + 24x - 48$

7.6 a) $y = f(x) = 3(x - 2)^2 - 4$

b) $y = f(x) = (x + 3)^2 - 9$

c) $y = f(x) = (x - 1)^2$

d) $y = f(x) = 2(x + 3)^2$

e) $y = f(x) = -2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$

f) $y = f(x) = x^2 + 1$

g) $y = f(x) = -\frac{1}{2}(x - 2)^2$

h) $y = f(x) = -4(x - 3)^2 - 7$

i) $y = f(x) = 2\left(x + \frac{1}{2}\right)^2 - \frac{49}{2}$

j) $y = f(x) = -\left(x - \frac{1}{4}\right)^2 + \frac{49}{16}$

7.7 a) i) $V(-3|2)$

ii) parabola opens upwards

b) i) $V\left(-\frac{3}{2}|-\frac{5}{8}\right)$

ii) parabola opens upwards

c) i) $V(2|1)$

ii) parabola opens downwards

d) i) $V(3|-1)$

ii) parabola opens downwards

e) i) $V\left(-3|-\frac{1}{4}\right)$

ii) parabola opens upwards

f) i) $V\left(\frac{3}{2}|\frac{5}{2}\right)$

ii) parabola opens downwards

7.8 a) 4th statement

b) 3rd statement

c) 1st statement