

Exercises 7 Quadratic function and equations Quadratic function

Objectives

- be able to graph a quadratic function out of the vertex form of its equation.
- be able to determine the position of the vertex of a parabola out of the vertex form of the equation of the corresponding quadratic function.
- be able to convert the vertex form of the equation of a quadratic function into the general form.
- know, understand, and be able to apply the method of completing the square.
- be able to convert the general form of the equation of a quadratic function into the vertex form.

Problems

7.1 Look at the easiest possible quadratic function:

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto y = f(x) = x^2 \end{aligned}$$

- a) Establish a table of values of f for the interval $-4 \leq x \leq 4$.
- b) Draw the graph of f in the interval $-4 \leq x \leq 4$ into a Cartesian coordinate system.

7.2 Generally, the equation of a quadratic function can be written in the so-called vertex form below:

$$\begin{aligned} f: D &\rightarrow \mathbb{R} && (D \subseteq \mathbb{R}) \\ x &\mapsto y = f(x) = a(x - u)^2 + v && (a \in \mathbb{R} \setminus \{0\}, u \in \mathbb{R}, v \in \mathbb{R}) \end{aligned}$$

In problems a) to d) ...

- i) ... sketch the graphs of the functions f_0 , f_1 , and f_2 into one coordinate system.
- ii) ... describe the influence of the parameters u , v , and a on the graph of the quadratic function.

a) Variation of parameter u (in all three cases below: $a = 1, v = 0$)

$$\begin{aligned} u = 0 : & & y = f_0(x) &= x^2 \\ u = 2 : & & y = f_1(x) &= (x - 2)^2 \\ u = -1 : & & y = f_2(x) &= (x + 1)^2 \end{aligned}$$

b) Variation of parameter v (in all three cases below: $a = 1, u = 0$)

$$\begin{aligned} v = 0 : & & y = f_0(x) &= x^2 \\ v = 3 : & & y = f_1(x) &= x^2 + 3 \\ v = -2 : & & y = f_2(x) &= x^2 - 2 \end{aligned}$$

c) Variation of parameter a (in all three cases below: $u = 0, v = 0$)

$$\begin{aligned} a = 1 : & & y = f_0(x) &= x^2 \\ a = 2 : & & y = f_1(x) &= 2x^2 \\ a = -2 : & & y = f_2(x) &= -2x^2 \end{aligned}$$

d) Variation of parameter a (in all three cases below: $u = 0, v = 0$)

$$\begin{aligned} a = 1 : & & y = f_0(x) &= x^2 \\ a = \frac{1}{2} : & & y = f_1(x) &= \frac{1}{2}x^2 \\ a = -\frac{1}{2} : & & y = f_2(x) &= -\frac{1}{2}x^2 \end{aligned}$$

7.3 For each quadratic function $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto y = f(x)$ in a) to h) ...

- i) ... state the parameters a, u, and v.
- ii) ... state the coordinates of the vertex of the graph.
- iii) ... state whether the parabola, i.e. the graph of the function, opens upwards or downwards.
- iv) ... graph the function.

a)	$y = f(x) = (x + 2)^2$	b)	$y = f(x) = -3x^2$
c)	$y = f(x) = 2x^2 - 1$	d)	$y = f(x) = -(x - 3)^2 + 4$
e)	$y = f(x) = \frac{1}{2}(x + 3)^2 + 2$	f)	$y = f(x) = -2(x - 1)^2 + 5$
g)	$y = f(x) = \frac{5}{2} - \left(x - \frac{1}{2}\right)^2$	h)	$y = f(x) = -\frac{1}{2} - 3(2 - x)^2$

7.4 The equation of a quadratic function can be written in the two forms below:

$$y = f(x) = ax^2 + bx + c \quad \text{general form}$$

$$y = f(x) = a(x - u)^2 + v \quad \text{vertex form}$$

- a) Verify that the vertex form of the equation can always be converted into the general form.
- b) Assume that the values of the parameters a, u, and v are known. Use the result in a) to determine the values of the parameters b and c out of a, u, and v.

7.5 The equation of a quadratic function f is written in the vertex form. Determine the general form of the equation:

a)	$y = f(x) = 2(x - 3)^2 + 4$	b)	$y = f(x) = -(x + 2)^2 - 3$
c)	$y = f(x) = x^2 + 5$	d)	$y = f(x) = -3(x - 4)^2$

7.6 Convert the given equation of a quadratic function into the vertex form by completing the square:

a)	$y = f(x) = 3x^2 - 12x + 8$	b)	$y = f(x) = x^2 + 6x$
c)	$y = f(x) = x^2 - 2x + 1$	d)	$y = f(x) = 2x^2 + 12x + 18$
e)	$y = f(x) = -2x^2 - 6x - 2$	f)	$y = f(x) = x^2 + 1$
g)	$y = f(x) = -\frac{1}{2}x^2 + 2x - 2$	h)	$y = f(x) = -4x^2 + 24x - 43$
i)	$y = f(x) = 2(x - 3)(x + 4)$	j)	$y = f(x) = x + 3 - \left(x + \frac{1}{2}\right)x$

7.7 For the graphs of the quadratic functions $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto y = f(x)$ in a) to f) ...

- i) ... determine the coordinates of the vertex.
- ii) ... state whether the parabola opens upwards or downwards.

a)	$y = f(x) = 2x^2 + 12x + 20$	b)	$y = f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{1}{2}$
c)	$y = f(x) = 12x - 3x^2 - 11$	d)	$y = f(x) = x(-0.2x + 1.2) - 2.8$
e)	$y = f(x) = \frac{17 + 12x + 2x^2}{4}$	f)	$y = f(x) = 7x(3 - x) - 13.25$

7.8 Decide which statements are true or false. Put a mark into the corresponding box.
In each problem a) to c), exactly one statement is true.

a) The graph of a quadratic function ...

- ... always intersects the x-axis in two points.
- ... opens downwards if it has no point in common with the x-axis.
- ... touches the x-axis if there is only one vertex.
- ... is always a parabola.

b) f is a linear function, and g is a quadratic function. It can be concluded that the graphs of f and g ...

- ... have no points in common.
- ... intersect only if the slope of f is not equal to zero.
- ... cannot have more than two points in common.
- ... have at least one point in common.

c) The vertex form of the equation of a quadratic function ...

- ... is equal to the general form if the vertex of the graph is on the y-axis.
- ... can be obtained from the general form by multiplying out all the terms.
- ... does not exist if the graph opens downwards.
- ... only depends on the position of the vertex.

Answers

7.1 see theory

7.2 see theory

- 7.3 a) i) $a = 1, u = -2, v = 0$
ii) $V(-2|0)$
iii) parabola opens upwards
iv) ...
- b) i) $a = -3, u = 0, v = 0$
ii) $V(0|0)$
iii) parabola opens downwards
iv) ...
- c) i) $a = 2, u = 0, v = -1$
ii) $V(0|-1)$
iii) parabola opens upwards
iv) ...
- d) i) $a = -1, u = 3, v = 4$
ii) $V(3|4)$
iii) parabola opens downwards
iv) ...
- e) i) $a = \frac{1}{2}, u = -3, v = 2$
ii) $V(-3|2)$
iii) parabola opens upwards
iv) ...
- f) i) $a = -2, u = 1, v = 5$
ii) $V(1|5)$
iii) parabola opens downwards
iv) ...
- g) i) $a = -1, u = \frac{1}{2}, v = \frac{5}{2}$
ii) $V\left(\frac{1}{2}|\frac{5}{2}\right)$
iii) parabola opens downwards
iv) ...

- h) i) $a = -3, u = 2, v = -\frac{1}{2}$
 ii) $V\left(2 \mid -\frac{1}{2}\right)$
 iii) parabola opens downwards
 iv) ...
- 7.4 a) $y = f(x) = a(x - u)^2 + v = \dots = ax^2 - 2aux + au^2 + v = ax^2 + (-2au)x + (au^2 + v)$
 Hints:
 - Expand the term $(x - u)^2$.
 - Simplify the whole expression.
 b) $b = -2au$
 $c = au^2 + v$
 Hint:
 - Compare the resulting expression in a) with the general form $ax^2 + bx + c$.
- 7.5 a) $y = f(x) = 2x^2 - 12x + 22$
 b) $y = f(x) = -x^2 - 4x - 7$
 c) $y = f(x) = x^2 + 5$
 Notice:
 - This is both the general and the vertex form of the equation.
 d) $y = f(x) = -3x^2 + 24x - 48$
- 7.6 a) $y = f(x) = 3(x - 2)^2 - 4$ b) $y = f(x) = (x + 3)^2 - 9$
 c) $y = f(x) = (x - 1)^2$ d) $y = f(x) = 2(x + 3)^2$
 e) $y = f(x) = -2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2}$ f) $y = f(x) = x^2 + 1$
 g) $y = f(x) = -\frac{1}{2}(x - 2)^2$ h) $y = f(x) = -4(x - 3)^2 - 7$
 i) $y = f(x) = 2\left(x + \frac{1}{2}\right)^2 - \frac{49}{2}$ j) $y = f(x) = -\left(x - \frac{1}{4}\right)^2 + \frac{49}{16}$
- 7.7 a) i) $V(-3 \mid 2)$ b) i) $V\left(-\frac{3}{2} \mid -\frac{5}{8}\right)$
 ii) parabola opens upwards ii) parabola opens upwards
 c) i) $V(2 \mid 1)$ d) i) $V(3 \mid -1)$
 ii) parabola opens downwards ii) parabola opens downwards
 e) i) $V\left(-3 \mid -\frac{1}{4}\right)$ f) i) $V\left(\frac{3}{2} \mid \frac{5}{2}\right)$
 ii) parabola opens upwards ii) parabola opens downwards
- 7.8 a) 4th statement
 b) 3rd statement
 c) 1st statement