Review exercises 2 Differential calculus, integral calculus

Problems

R2.1 Decide whether the following statements are true or false:

a) "The derivative (derived function) of a function is a function."

b) "The derivative (rate of change) of a function at a particular position is a number."

c) "The function f has a relative maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."

d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."

e) "If $g' = f$, then g is an antiderivative of f."

f) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."

g) "f with $f(x) = 3x$ has infinitely many antiderivatives."

h) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ at x_0 for the following functions f:

a) $f(x) = 4x^2(x^2 - 1)$

i) $x_0 = 0$ ii) $x_0 = -1$

b) $f(x) = (- 3x^2 + 2x - 1)\cdot e^x$

i) $x_0 = 0$ ii) $x_0 = -2$

c) $f(x) = (x^2 + 2)\cdot e^{-3x}$

i) $x_0 = 1$ ii) $x_0 = -\frac{1}{3}$

R2.3 For the given cost function C(x) and revenue function R(x) determine ...

i) ... the marginal cost function C'(x).

ii) ... the marginal revenue function R'(x).

iii) ... the marginal profit function P'(x).

a) $C(x) = 200 + 40x$ $R(x) = 60x$

b) $C(x) = 100 + 20x + 5x^2$ $R(x) = 100x - 2x^2$

c) $C(x) = 50 + 20x^2 + 3e^{4x}$ $R(x) = 200x - e^{-4x^2}$

R2.4 For each function, determine …

i) … the relative maxima and minima.

ii) … the points of inflection.

a) $f(x) = 2x^3 - 9x^2 + 12x - 1$

b) f(x) as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$$R(x) = 36x - 0.01x^2$$

Determine the maximum revenue if production is limited to at most 1500 units.

R2.6 If the total cost function for a product is

$$C(x) = 100 + x^2$$

producing how many units x will result in a minimum average cost? Determine the minimum average cost.

R2.7 A firm can produce only 1000 units per month. The monthly total cost (in CHF) ist given by

$$C(x) = 300 + 200x$$

where x is the number produced. If the total revenue (in CHF) is given by

$$R(x) = 250x - \frac{1}{100}x^2$$

how many items should the firm produce for maximum profit? Determine the maximum profit.

R2.8 Determine the indefinite integrals below:

a) $\int (x^4 - 3x^3 - 6)\ dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\ dx$

R2.9 The equation of the third derivative f ''' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Determine the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.10 If the marginal cost (in CHF) for producing a product is $C'(x) = 5x + 10$, with a fixed cost of 800 CHF, what will be the cost of producing 20 units?

R2.11 A certain firm's marginal cost C'(x) and the derivative of the average revenue $\overline{R}'(x)$ are given as follows:

$$C'(x) = 6x + 60$$

$$\overline{R}'(x) = -1$$

The total cost and revenue of the production of 10 items are 1000 CHF and 1700 CHF, respectively.

How many units will result in a maximum profit? Determine the maximum profit.

R2.12 The demand function (price in CHF) for a product is

$$p = f(x) = 49 - x^2$$

and the supply function (price in CHF) is

$$p = g(x) = 4x + 4$$

Determine the equilibrium point and both the consumer's and the producer's surplus there.

R2.13 (see next page)

R2.13 The demand function (price in CHF) for a product is

$$p = f(x) = 110 - ax^2$$

and the supply function (price in CHF) is

$$p = g(x) = 2 - \frac{6}{5}x + bx^2$$

with unknown parameters a and b. The equilibrium price is 10 CHF, and the producer's surplus is 73.33 CHF.

Determine the two unknown parameters a and b.

Answers

R2.1

	a)	true	b)	true	c)	false
	d)	true	e)	true	f)	false
	g)	true	h)	true		

R2.2 a) $f'(x) = 16x^3 - 8x$
$f''(x) = 48x^2 - 8$

i) $f(0) = 0$ $\quad f'(0) = 0$ $\quad f''(0) = -8$

ii) $f(-1) = 0$ $\quad f'(-1) = -8$ $\quad f''(-1) = 40$

b) $f'(x) = (-3x^2 - 4x + 1)\cdot e^x$
$f''(x) = (-3x^2 - 10x - 3)\cdot e^x$

i) $f(0) = -1$ $\quad f'(0) = 1$ $\quad f''(0) = -3$

ii) $f(-2) = -17\cdot e^{-2} = -2.300\ldots$
$f'(-2) = -3\cdot e^{-2} = -0.406\ldots$
$f''(-2) = 5\cdot e^{-2} = 0.676\ldots$

c) $f'(x) = (-3x^2 + 2x - 6)\cdot e^{-3x}$
$f''(x) = (9x^2 - 12x + 20)\cdot e^{-3x}$

i) $f(1) = 3\cdot e^{-3} = 0.149\ldots$
$f'(1) = -7\cdot e^{-3} = -0.348\ldots$
$f''(1) = 17\cdot e^{-3} = 0.846\ldots$

ii) $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\ldots$
$f'\left(-\frac{1}{3}\right) = -7e = -19.027\ldots$
$f''\left(-\frac{1}{3}\right) = 25e = 67.957\ldots$

R2.3 a) i) $C'(x) = 40$ $\quad$ ii) $R'(x) = 60$

iii) $P'(x) = 20$

b) i) $C'(x) = 20 + 10x$ $\quad$ ii) $R'(x) = 100 - 4x$

iii) $P'(x) = 80 - 14x$

c) i) $C'(x) = 40x + 12e^{4x}$ $\quad$ ii) $R'(x) = 200 + 8x\, e^{-4x^2}$

iii) $P'(x) = 200 - 40x - 12e^{4x} + 8x\, e^{-4x^2}$

R2.4 a) $f(x) = 2x^3 - 9x^2 + 12x - 1$
$f'(x) = 6x^2 - 18x + 12$
$f''(x) = 12x - 18$
$f'''(x) = 12$

i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
$f''(x_1) = -6 < 0$ $\quad \Rightarrow \quad$ relative maximum at $x_1 = 1$
$f''(x_2) = 6 > 0$ $\quad \Rightarrow \quad$ relative minimum at $x_2 = 2$

ii) $f''(x) = 0$ at $x_3 = \frac{3}{2}$
$f'''(x_3) = 12 \neq 0$ $\quad \Rightarrow \quad$ point of inflection at $x_3 = \frac{3}{2}$

b) $f(x) = 4x^2(x^2 - 1)$
$f'(x) = 16x^3 - 8x = 8x(2x^2 - 1)$
$f''(x) = 48x^2 - 8 = 8(6x^2 - 1)$
$f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
$f''(x_1) = -8 < 0 \quad \Rightarrow \quad$ relative maximum at $x_1 = 0$
$f''(x_2) = 16 > 0 \quad \Rightarrow \quad$ relative minimum at $x_2 = \frac{1}{\sqrt{2}}$
$f''(x_3) = 16 > 0 \quad \Rightarrow \quad$ relative minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f''(x) = 0$ at $x_3 = \frac{1}{\sqrt{6}}$
$f'''(x_3) = \frac{96}{\sqrt{6}} \neq 0 \quad \Rightarrow \quad$ point of inflection at $x_3 = \frac{1}{\sqrt{6}}$

R2.5 **Relative** maximum at $x = 1800$ lies outside the possible interval $0 \leq x \leq 1500$
$R(1500) = 31'500$ CHF $> R(0) = 0$ CHF
$\Rightarrow$ $R = 31'500$ CHF is the **absolute** maximum revenue at $x = 1500$.

R2.6 $\overline{C}(x) = \frac{C(x)}{x} = \frac{100}{x} + x$
$\overline{C}(x)$ has a **relative** minimum at $x_1 = 10$
$\overline{C}(10) = 20$ CHF
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no relative maximum
$\Rightarrow$ $\overline{C} = 20$ CHF is the **absolute** minimum average cost at $x = 10$.

R2.7 $P(x) = R(x) - C(x) = -\frac{1}{100}x^2 + 50x - 300$
$P(x)$ has a **relative** maximum at $x_1 = 2500$. This is outside the possible interval $0 \leq x \leq 1000$
$P(1000) = 39'700$ CHF $> P(0) = -300$ CHF
$\Rightarrow$ $P = 39'700$ CHF is the **absolute** maximum profit at the endpoint $x = 1000$.

R2.8 a) $\int (x^4 - 3x^3 - 6)\ dx = \frac{x^5}{5} - \frac{3x^4}{4} - 6x + C$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right)\ dx = \frac{x^7}{14} + \frac{2}{9x^3} + C$

R2.9 $f(x) = \frac{x^4}{8} + \frac{x^3}{6} + x + 2$

R2.10 $C(20) = 2000$ CHF

Hint:
- First, determine the cost function $C(x)$ $\Rightarrow$ $C(x) = \frac{5}{2}x^2 + 10x + 800$

R2.11 $P = 800$ CHF is the absolute maximum profit at $x = 15$ units.

Hints:
- Determine the cost function $C(x)$ $\Rightarrow$ $C(x) = 3x^2 + 60x + 100$
- Determine the average revenue function $\overline{R}(x)$ $\Rightarrow$ $\overline{R}(x) = -x + C$
- Determine the revenue function $R(x)$ $\Rightarrow$ $R(x) = -x^2 + 180x$
- Determine the profit function $P(x)$ $\Rightarrow$ $P(x) = -4x^2 + 120x - 100$
- Determine the relative maximum of the profit function $P(x)$.
- Check if the relative maximum is the absolute maximum.

R2.12 Equilibrium quantity $x = 5$
Equilibrium price $p = 24$ CHF
Consumer's surplus $CS = 83.33$ CHF
Producer's surplus $PS = 50$ CHF

R2.13 $a = 1$
$b = 0.2$