Review exercises 2 — Differential calculus, integral calculus

Problems

R2.1 Decide whether the statements below are true or false:

a) "The derivative (derived function) of a function is a function."

b) "The derivative (rate of change) of a function at a particular position is a number."

c) "The function f has a local maximum at $x = x_1$ if $f'(x_1) = 0$ and $f''(x_1) > 0$."

d) "If $f''(x_2) = 0$ and $f'''(x_2) < 0$, then the function f has a point of inflection at $x = x_2$."

e) "If $g' = f$, then g is an antiderivative of f."

f) "f with $f(x) = 2x + 20$ is an antiderivative of g with $g(x) = x^2$."

g) "f with $f(x) = 3x$ has infinitely many antiderivatives."

h) "The indefinite integral of a function is a set of functions."

R2.2 Determine the value $f(x_0)$, the first derivative $f'(x_0)$, and the second derivative $f''(x_0)$ of the function f at the position x_0:

a) $f(x) = 4x^2(x^2 - 1)$ $\qquad x_0 = -1$

b) $f(x) = (-3x^2 + 2x - 1)\cdot e^x$ $\qquad x_0 = -2$

c) $f(x) = (x^2 + 2)\cdot e^{-3x}$ $\qquad x_0 = -\frac{1}{3}$

R2.3 For the given cost function C(x) and revenue function R(x) determine ...

i) ... the marginal cost function C'(x).

ii) ... the marginal revenue function R'(x).

iii) ... the marginal profit function P'(x).

a) $C(x) = (40x + 200)$ CHF $\qquad R(x) = 60x$ CHF

b) $C(x) = (5x^2 + 20x + 100)$ CHF $\qquad R(x) = (-2x^2 + 100x)$ CHF

c) $C(x) = (20x^2 + 50 + 3e^{4x})$ CHF $\qquad R(x) = \left(200x - e^{-4x^2}\right)$ CHF

R2.4 For the function f, determine …

i) … the local maxima and minima.

ii) … the points of inflection.

a) $f(x) = 2x^3 - 9x^2 + 12x - 1$

b) f(x) as in R2.2 a)

R2.5 The total revenue function for a commodity is given by

$$R(x) = (-0.01x^2 + 36x) \text{ CHF}$$

Determine the maximum revenue if production is limited to at most 1500 units.

R2.6 If the total cost function for a product is

$$C(x) = (x^2 + 100) \text{ CHF}$$

producing how many units x will result in a minimum average cost? Determine the minimum average cost.

R2.7 A firm can produce 1000 units per month only. The monthly total cost is given by

$$C(x) = (200x + 300) \text{ CHF}$$

where x is the number produced. If the total revenue is given by

$$R(x) = \left(-\frac{1}{100}x^2 + 250x\right) \text{ CHF}$$

how many items should the firm produce for a maximum profit? Determine the maximum profit.

R2.8 Determine the indefinite integrals below:

a) $\int (x^4 - 3x^3 - 6)\, dx$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) dx$

R2.9 The equation of the third derivative f ''' of a function f is given as follows:

$$f'''(x) = 3x + 1$$

Determine the equation of the function f such that $f''(0) = 0$, $f'(0) = 1$, $f(0) = 2$

R2.10 If the marginal cost for producing a product is $C'(x) = (5x + 10)$ CHF, with a fixed cost of 800 CHF, what will be the cost of producing 20 units?

R2.11 A certain firm's marginal cost C'(x) and the derivative of the average revenue $\overline{R}'(x)$ are given as follows:

$$C'(x) = (6x + 60) \text{ CHF}$$

$$\overline{R}'(x) = -1 \text{ CHF}$$

The total cost and revenue of the production of 10 items are 1000 CHF and 1700 CHF, respectively.

How many units will result in a maximum profit? Determine the maximum profit.

R2.12 The supply function for a product is

$$p = f_s(x) = (4x + 4) \text{ CHF}$$

and the demand function is

$$p = f_d(x) = (-x^2 + 49) \text{ CHF}$$

Determine the equilibrium point and both the consumer’s and the producer’s surplus there.

R2.13 (see next page)

R2.13 The supply function for a product is

$$p = f_s(x) = \left(ax^2 - \frac{6}{5}x + 2\right) \text{ CHF}$$

and the demand function is

$$p = f_d(x) = (-bx^2 + 110) \text{ CHF}$$

with unknown parameters a and b. The equilibrium price is 10 CHF, and the producer's surplus is 73.33 CHF (rounded).

Determine the two unknown parameters a and b.

Hint:
- Use the unrounded value $\left(73 + \frac{1}{3}\right)$ CHF = $\frac{220}{3}$ CHF for the producer's surplus.

Answers

R2.1

a) true

b) true

c) false

d) true

e) true

f) false

g) true

h) true

R2.2

a)
$f(x) = 4x^2(x^2 - 1)$ — $f(-1) = 0$
$f'(x) = 16x^3 - 8x$ — $f'(-1) = -8$
$f''(x) = 48x^2 - 8$ — $f''(-1) = 40$

b)
$f(x) = (-3x^2 + 2x - 1)\cdot e^x$ — $f(-2) = -17\cdot e^{-2} = -2.300\ldots$
$f'(x) = (-3x^2 - 4x + 1)\cdot e^x$ — $f'(-2) = -3\cdot e^{-2} = -0.406\ldots$
$f''(x) = (-3x^2 - 10x - 3)\cdot e^x$ — $f''(-2) = 5\cdot e^{-2} = 0.676\ldots$

c)
$f(x) = (x^2 + 2)\cdot e^{-3x}$ — $f\left(-\frac{1}{3}\right) = \frac{19}{9}e = 5.738\ldots$
$f'(x) = (-3x^2 + 2x - 6)\cdot e^{-3x}$ — $f'\left(-\frac{1}{3}\right) = -7e = -19.027\ldots$
$f''(x) = (9x^2 - 12x + 20)\cdot e^{-3x}$ — $f''\left(-\frac{1}{3}\right) = 25e = 67.957\ldots$

R2.3

a)
i) $C'(x) = 40$ CHF
ii) $R'(x) = 60$ CHF
iii) $P'(x) = R'(x) - C'(x) = 20$ CHF

b)
i) $C'(x) = (10x + 20)$ CHF
ii) $R'(x) = (-4x + 100)$ CHF
iii) $P'(x) = R'(x) - C'(x) = (-14x + 80)$ CHF

c)
i) $C'(x) = (40x + 12e^{4x})$ CHF
ii) $R'(x) = \left(200 + 8x\,e^{-4x^2}\right)$ CHF
iii) $P'(x) = R'(x) - C'(x) = \left(-40x + 200 - 12e^{4x} + 8x\,e^{-4x^2}\right)$ CHF

R2.4

a)
$f(x) = 2x^3 - 9x^2 + 12x - 1$
$f'(x) = 6x^2 - 18x + 12$
$f''(x) = 12x - 18$
$f'''(x) = 12$

i) $f'(x) = 0$ at $x_1 = 1$ and $x_2 = 2$
$f''(x_1) = -6 < 0$ $\Rightarrow$ local maximum at $x_1 = 1$
$f''(x_2) = 6 > 0$ $\Rightarrow$ local minimum at $x_2 = 2$

ii) $f''(x) = 0$ at $x_3 = \frac{3}{2}$
$f'''(x_3) = 12 \neq 0$ $\Rightarrow$ point of inflection at $x_3 = \frac{3}{2}$

b) (see next page)

b) $f(x) = 4x^2(x^2 - 1) = 4x^4 - 4x^2$
$f'(x) = 16x^3 - 8x$
$f''(x) = 48x^2 - 8$
$f'''(x) = 96x$

i) $f'(x) = 0$ at $x_1 = 0$, $x_2 = \frac{1}{\sqrt{2}}$, and $x_3 = -\frac{1}{\sqrt{2}}$
$f''(x_1) = -8 < 0$ ⇒ local maximum at $x_1 = 0$
$f''(x_2) = 16 > 0$ ⇒ local minimum at $x_2 = \frac{1}{\sqrt{2}}$
$f''(x_3) = 16 > 0$ ⇒ local minimum at $x_3 = -\frac{1}{\sqrt{2}}$

ii) $f''(x) = 0$ at $x_4 = \frac{1}{\sqrt{6}}$ and $x_5 = -\frac{1}{\sqrt{6}}$
$f'''(x_4) = \frac{96}{\sqrt{6}} \neq 0$ ⇒ point of inflection at $x_4 = \frac{1}{\sqrt{6}}$
$f'''(x_5) = -\frac{96}{\sqrt{6}} \neq 0$ ⇒ point of inflection at $x_5 = -\frac{1}{\sqrt{6}}$

R2.5 **Local** maximum at $x = 1800$ lies outside the possible interval $0 \leq x \leq 1500$.
$R(1500) = 31'500$ CHF $> R(0) = 0$ CHF
⇒ $R = 31'500$ CHF is the **global** maximum revenue at $x = 1500$.

R2.6 $\overline{C}(x) = \frac{C(x)}{x} = \left(x + \frac{100}{x}\right)$ CHF
$\overline{C}(x)$ has a **local** minimum at $x_1 = 10$.
$\overline{C}(10) = 20$ CHF
$\overline{C}(x) > \overline{C}(x_1)$ if $x \neq x_1$ as there is no local maximum.
⇒ $\overline{C} = 20$ CHF is the **global** minimum average cost at $x = 10$.

R2.7 $P(x) = R(x) - C(x) = \left(-\frac{1}{100}x^2 + 50x - 300\right)$ CHF
$P(x)$ has a **local** maximum at $x_1 = 2500$. This is outside the possible interval $0 \leq x \leq 1000$.
$P(1000) = 39'700$ CHF $> P(0) = -300$ CHF
⇒ $P = 39'700$ CHF is the **global** maximum profit at the endpoint $x = 1000$.

R2.8 a) $\int (x^4 - 3x^3 - 6)\, dx = \frac{1}{5}x^5 - \frac{3}{4}x^4 - 6x + C$

b) $\int \left(\frac{1}{2}x^6 - \frac{2}{3x^4}\right) dx = \frac{1}{14}x^7 + \frac{2}{9x^3} + C$

R2.9 $f(x) = \frac{1}{8}x^4 + \frac{1}{6}x^3 + x + 2$

R2.10 $C(20) = 2000$ CHF

Hint:
- First, determine the cost function $C(x)$ ⇒ $C(x) = \left(\frac{5}{2}x^2 + 10x + 800\right)$ CHF

R2.11 (see next page)

R2.11 P = 800 CHF is the global maximum profit at x = 15 units.

Hints:
- Determine the cost function C(x) $\Rightarrow$ C(x) = ($3x^2$ + 60x + 100) CHF
- Determine the average revenue function $\overline{R}(x)$ $\Rightarrow$ $\overline{R}(x)$ = (- x + C) CHF
- Determine the revenue function R(x) $\Rightarrow$ R(x) = (- x^2 + 180x) CHF
- Determine the profit function P(x) $\Rightarrow$ P(x) = (- $4x^2$ + 120x - 100) CHF
- The profit function P(x) is a quadratic function.
- Think of the graph of the profit function when determining the gobal maximum.

R2.12 Equilibrium quantity
x = 5

Equilibrium price
p = 24 CHF

Consumer's surplus
CS = 83.33 CHF (rounded)

Producer's surplus
PS = 50 CHF

R2.13 $a = \frac{1}{5}$
b = 1